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LETTER TO THE EDITOR

Stochastic Schrödinger and Heisenberg equations: a martingale problem in quantum stochastic processes

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Abstract. The relation between a quantum dynamical semigroup and the associated quantum stochastic processes can be formulated by an analogy in operator algebras to the martingale problem of Stroock and Varadhan for classical diffusions. It is shown that under simple circumstances the formulation yields the explicit solution. We give three such examples.

Quantum dynamical semigroup (QDS) is the notion which was introduced in the last decade for the non-commutative operator-algebra version of classical diffusion: some important papers in the subject are in theoretical quantum optics by several authors: Kossakowski (1972), Hepp and Lieb (1973), Ingarden and Kossakowski (1975), Lindblad (1976), Gorini *et al* (1976) and others (cf Davies 1976). Recently, considerable effort has been made to describe such quantum diffusions by means of stochastic differential equations (SDE) and their integrals that are operator valued (Hudson and Streater 1981a, b, Barnett *et al* 1982, Streater 1982a, Hudson and Parthasarathy 1982).

In this report, we point out that the basic question, namely to find the SDE associated with a given QDS (in terms of its generator), can be formulated concisely by the so-called 'martingale problem' in algebras, that is to say, by the quantum analogue of the martingale problem due to Stroock and Varadhan (1969) for classical diffusions. We also note that under some special circumstances the problem can be solved within the classical framework in terms of what may be called 'stochastic Heisenberg (or, its counterpart of Schrödinger) equation' aided by the stochastic calculus of Itô. Explicit examples in physics which we show here are as follows:

(i) harmonic oscillator,

(ii) Pauli spin,

whose frequencies are modulated very fast, and

(iii) diffusion for Fermi fields.

Consider a (one-dimensional) classical diffusion process x_t $(=x(t,\omega))$ subject to the Itô SDE

$$dx_t = b(x_t) + \sigma(x_t)\, dB(t). \tag{1}$$

A smooth function f of the process x_t, $f(x_t)$, can be related to SDE (1) via the Itô formula as

$$f(x_t) - f(x_0) - \int_0^t Lf(x_\tau)\, d\tau = M[f]_t \qquad (0 \leq t < \infty) \tag{2}$$

where L is the generator $(=b(x)\,\partial/\partial x+\frac{1}{2}\sigma^2(x)\,\partial^2/\partial x^2)$ of the contraction semigroup that represents the diffusion of solutions to (1), and $M[f]_t$ denotes a stochastic integral. The force $M[f]_t$ has the martingale property

$$E(M[f]_t|\mathcal{F}_s)=M[f]_s, \qquad 0\leq s\leq t, \tag{3}$$

with respect to the filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, generated by the process x_t, as well as with respect to the filtration generated by B_t, the standard Brownian motion. (Here, $\mathcal{F}_t$ is the σ-ring of Borel sets generated by $\{x_s;\, s\leq t\}$.) The classical contraction semigroup $\{T_t=\mathrm{e}^{tL};\, 0\leq t<\infty\}$ inherent in (1) can be deduced naturally from the conditional expectation $E(\cdot|\mathcal{F}_{t=0})$, i.e. from

$$f_t\equiv E\{f(x_t)|\mathcal{F}_0\}$$

as an element of the $L^2(R)$-space of functions of $x\in\mathbb{R}$, on which we have

$$f_t=T_tf_{t=0}; \qquad T_{t+s}=T_t\cdot T_s, \qquad T_0=1, \qquad \|T_t\|\leq 1, \tag{4}$$

since $f_t-f_0-\int_0^t Lf_\tau\,\mathrm{d}\tau=0$ implies that $\mathrm{d}f_t/\mathrm{d}t=Lf_t$ holds in a weak sense.

What is important for the martingale structure of diffusion is the converse to the above. Suppose that a classical semigroup (4) defined on the function space with a diffusion operator L as its generator is given. Since L provides us with information only about the conditionally averaged sample paths, one asks if the complete knowledge about the sample paths could be recovered from the semigroup generator L alone. This question, to find the SDE (1) given only the left-hand side of (2), was answered by Stroock and Varadhan: these authors were thus able to enlarge the class of the b- and σ-functions in SDE (1) which lead to a legitimate diffusion process.

Let us propose, therefore, a similar problem for quantum dynamical semigroups. That is, suppose we are given a generator of a QDS defined on an algebra, and ask ourselves: what is quantum mechanical 'sample space' that conforms to the given generator? Since we know that a quantum mechanical time evolution of the system variables (i.e. observables) is the Heisenberg motion expressed as a unitary evolution, $X\to U_t^*XU_t$ for $X\in\mathcal{B}(\mathcal{H}_s)$, the requirement must be†

$$\mathbb{E}\{U_t^*XU_t|\mathcal{F}_s\}=\mathrm{e}^{(t-s)L}(U_s^*XU_s) \qquad (0\leq s\leq t) \tag{5}$$

for the given QDS map e^{tL} defined on the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators X defined on the system Hilbert space $\mathcal{H}$. The reference family $\{\mathcal{F}_t;\, -\infty<t<\infty\}$ and the conditional expectation $\mathbb{E}\{\cdot|\mathcal{F}_s\}$, however, should be suitably reinterpreted so that the quantum stochastic process $X_t=U_t^*XU_t$ is general enough to include the operator (non-commutative) martingales (Streater 1982a). Or, from a more physical point of view, the underlying Hilbert space of the unitary evolution should be larger than $\mathcal{H}$: it should be $\mathcal{H}\otimes\Gamma$ where Γ is the noise space of the heat-bath variables.

We now show that there exists an elementary class of QDS generators for which the above martingale problem has an unambiguous formulation. Let $\mathcal{A}$ be a *-algebra generated by two elements A and A^* such that there exists a self-adjoint element $Z\in\mathcal{A}$ for which we have

$$[Z,A^*](=ZA^*-A^*Z)=A^*, \qquad [Z,A]=-A. \tag{6}$$

† To find a quantum (or, operator-valued) SDE of unitary evolution associated with a prototype generator by the name of 'martingale problem' has been proposed also by Parthasarathy (private communication 1983).

Then, the martingale problem in $\mathscr{A}$, the required equation (2) being now

$$X_t - X - \int_0^t LX_\tau \, \mathrm{d}\tau = M[X_\tau]_t \tag{7}$$

with

$$L = \mathrm{i}\omega[Z, \cdot] - \tfrac{1}{2}\gamma[Z, [Z, \cdot]], \qquad \gamma > 0, \tag{8}$$

has the following solution:

$$M[X_\tau]_t = \mathrm{i} \int_0^t [\sqrt{\gamma} Z, X_\tau] \, \mathrm{d}\beta(\tau), \tag{9}$$

where $\beta(t) \equiv B(t) - B(-t)$, $B(t)$: the standard Brownian motion for $t \in [0, \infty]$, and $X_t \equiv U_t^* X U_t$ is the Heisenberg motion with

$$U_t = \exp[-\mathrm{i}(\omega t + \sqrt{\gamma}\beta(t))Z]. \tag{10}$$

For this U_t, the relation (5) between the conditional expectation and the positive map given by the QDS holds. The unitary evolution U_t itself satisfies another martingale equation

$$U_t - 1 - \int_0^t (-\mathrm{i}\omega Z - \tfrac{1}{2}\gamma Z^2) U_\tau \, \mathrm{d}\tau = -\mathrm{i} \int_0^t \sqrt{\gamma} Z U_\tau \, \mathrm{d}\beta(\tau). \tag{11}$$

It may be noted that the quantum stochastic processes in this example are all induced by a single, classical Brownian motion $\beta(t)$: the filtration $\{\mathscr{F}_t\}$ is the usual increasing family of σ-rings; these may be associated with the abelian W^*-algebras $R_t = L^\infty(\Omega, \mu, \mathscr{F}_t)$, where Ω is Wiener space and μ is Wiener measure. (Γ is $L^2(\Omega, \mu, \mathscr{F}_\infty)$ in this case.) We also note that such random motions in a quantum system without thermal contact with a heat bath may still give rise to dissipation (the so-called 'random frequency modulation' (Hasegawa and Ezawa 1980)).

We first show that the unitary evolution operator U_t given by (10) (the unitarity is obvious, because $\beta(t)$ is real and Z is self-adjoint) satisfies the martingale equation (11); a fact which is equivalent to the martingale property of the modified exponential of a martingale: this can be deduced most clearly by using the symmetric stochastic chain rule (Itô and Watanabe 1976) as follows:

$$\begin{aligned} \mathrm{d}U_t &= (\partial U_t / \partial \beta) \circ \mathrm{d}\beta(t) - \mathrm{i}\omega Z U_t \, \mathrm{d}t \\ &= -\mathrm{i}\sqrt{\gamma} Z U_t \circ \mathrm{d}\beta(t) - \mathrm{i}\omega Z U_t \, \mathrm{d}t \\ &= -\mathrm{i}\sqrt{\gamma} Z U_t \, \mathrm{d}\beta(t) + \tfrac{1}{2}(-\mathrm{i}\sqrt{\gamma})^2 Z^2 U_t (\mathrm{d}\beta(t))^2 - \mathrm{i}\omega Z U_t \, \mathrm{d}t \\ &= -\mathrm{i}\sqrt{\gamma} Z U_t \, \mathrm{d}\beta(t) + (-\mathrm{i}\omega Z - \tfrac{1}{2}\gamma Z^2) U_t \, \mathrm{d}t. \end{aligned} \tag{12}$$

Thus, integrating both sides in the interval $\tau \in [0, t]$ and noting that $U_{t=0} = 1$, the relation (11) can be obtained, where the right side is a martingale. When regarded as a SDE for the operator U_t with initial value $U_{t=0} = 1$, it has the unique solution (10). This is because the stochastic calculus (including the symmetric chain rule) again enables us to manipulate thus:

$$U_t^{-1} \circ \mathrm{d}U_t = (\circ \, \mathrm{d}U_t) U_t^{-1} \, (= \mathrm{d} \log U_t) = -\mathrm{i}(\omega t + \sqrt{\gamma} \, \mathrm{d}\beta(t)) Z. \tag{13}$$

The validity of these manipulations depends crucially on the commutativity of Z with

U_t, and with this hypothesis the martingale problem

$$U_t - 1 - \int_0^t (-\mathrm{i}\omega Z - \tfrac{1}{2}\gamma Z^2) U_\tau \, \mathrm{d}\tau = M[U_\tau]_t$$

has the solution

$$M[U_\tau]_t = -\mathrm{i} \int_0^t \sqrt{\gamma} Z U_\tau \, \mathrm{d}\beta(\tau) \tag{14}$$

with

$$U_\tau = \exp[-\mathrm{i}(\omega\tau + \sqrt{\gamma}\beta(\tau))Z]. \tag{14a}$$

Next, we consider the Heisenberg motion $X \to U_t^* X U_t$ with the above unitary operator. The symmetric stochastic chain rule again enables us to write

$$\begin{aligned}
\mathrm{d}X_t &= U_t^*[(\mathrm{i}\omega\,\mathrm{d}t + \mathrm{i}\sqrt{\gamma} \circ \mathrm{d}\beta(t))Z, X]U_t \\
&= [(\mathrm{i}\omega\,\mathrm{d}t + \mathrm{i}\sqrt{\gamma} \circ \mathrm{d}\beta(t))Z, U_t^* X U_t] \\
&= [(\mathrm{i}\omega\,\mathrm{d}t + \mathrm{i}\sqrt{\gamma}\,\mathrm{d}\beta(t))Z, X_t] + \tfrac{1}{2}\sqrt{\gamma}\,\mathrm{d}\beta(t)[Z, \mathrm{d}X_t] \\
&= [\mathrm{i}\sqrt{\gamma}Z, X_t]\,\mathrm{d}\beta(t) + [\mathrm{i}\omega Z, X_t]\,\mathrm{d}t - \tfrac{1}{2}\gamma[Z,[Z,X_t]]\,\mathrm{d}t.
\end{aligned} \tag{15}$$

Thus, an integration of both sides of (15) in the interval of $\tau \in [0, t]$ with initial value $X_{t=0} = X$ yields (7), (8) and (9), and the fact that (9) yields the unique solution of the martingale problem (7) is, as before, the consequence of the Heisenberg motion $X_t = U_t^* X U_t$ to be the unique solution of SDE (15). Therefore, our remaining task is to show that the unitary evolution operator U_t given by (10) really satisfies the conditional expectation-positive map relation (5). For this purpose, let us re-express (15) as

$$\mathrm{d}X_t = L_M X_t\,\mathrm{d}\beta(t) + LX_t\,\mathrm{d}t \qquad (L_M = \mathrm{i}[\sqrt{\gamma}Z, \cdot\,] \text{ and } L \text{ in } (8)),$$

that is, the decomposition of the operator stochastic differential $\mathrm{d}X_t$ into the martingale part and the bounded-variation part. The Heisenberg motion X_t is then

$$X_t = \exp[\beta(t)L_M + tL]X = \exp[\beta(t)L_M](\mathrm{e}^{tL}X),$$

since L_M commutes with L. A more general connection of X_t with X_s at an earlier time $s < t$ may also be expressed as

$$X_t = U_{t,s}^* X_s U_{t,s} \tag{16}$$

where

$$U_{t,s} = \exp\{-\mathrm{i}[\omega(t-s) + \sqrt{\gamma}(\beta(t) - \beta(s))]Z\}, \tag{17}$$

so that the previous U_t is given by $U_{t,0}$. Therefore,

$$X_t = \exp[(\beta(t) - \beta(s))L_M](\mathrm{e}^{(t-s)L}X_s) \tag{18}$$

again because of the commutativity between L_M and L. Now, $\mathbb{E}\{\exp[(\beta(t) - \beta(s))L_M] \| \mathscr{F}_s\} = 1$ by virtue of the stochastic independence between the processes $\beta(t) - \beta(s)$ and $\beta(s)$, $s \leq t$, and hence the relation (5) follows.

It may be observed in the above that the necessary tools for the stochastic processes are available within the classical framework. Our results can be summarised by the two forms of SDE, i.e. the Schrödinger SDE and the Heisenberg SDE (but in terms of the so-called Fisk–Stratonovich symmetric differentials) as follows:

$$(\mathrm{S})\ \mathrm{i}\,\mathrm{d}U_t = \omega Z U_t\,\mathrm{d}t + \sqrt{\gamma} Z U_t \circ \mathrm{d}\beta(t),$$

$$(\mathrm{H})\ \mathrm{d}X_t = \mathrm{i}[\omega Z, X_t]\,\mathrm{d}t + \mathrm{i}[\sqrt{\gamma} Z, X_t] \circ \mathrm{d}\beta(t).$$

The general solution of (S), $U_{t,s}$ given by (17), constitutes a two-parameter family of unitary operators, and satisfies the time-reversal symmetry (symmetric stochastic integrals (Itô 1976))

$$U^*_{t,s} = U_{s,t}. \tag{19}$$

We now show two elementary examples of the above.

(i) Harmonic oscillator: $Z = a^*a$, $A = a$ (a: boson annihilation operator). The martingale problem for this system may be compared with the question of Senitzky (1960), i.e. to find out an operator Langevin equation for a damped harmonic oscillator that retains the canonical commutation relation. Our answer here is, however, basically different from the one which satisfies the KMS condition (Streater 1982b).

(ii) Pauli spin: $Z = \frac{1}{2}\sigma^z$, $A = \sigma^x - \mathrm{i}\sigma^y$. The Heisenberg motion $\sigma^\pm\ (= \sigma^x \pm \mathrm{i}\sigma^y) \to U_t^* \sigma^\pm U_t$ is shown to give rise to a Larmor precession of the spin in an inhomogeneous magnetic field with the Lorentzian (Cauchy) distribution, familiar in spin echoes (Lindblad 1980): the semigroup map with the generator (8) here is equivalent to the average of the Larmor precession over the Cauchy field-distribution.

Our third example depends on the theory of the Itô–Clifford integral (Barnett *et al* 1982). Let the 'system' Hilbert space be $\mathscr{H} = \mathbb{C}^{2^n}$, identified with Fermion Fock space over n Clifford elements $\psi_1, \dots, \psi_n$. Let $\Psi_1(t), \dots, \Psi_n(t)$ be n independent anticommuting copies of the Itô–Clifford process: they act on the noise space Γ. Let $F_{jk}(X_1, \dots, X_n, t)$ $(j, k = 1, 2, \dots, n)$ be self-adjoint, even, time-dependent functions of n self-adjoint operators $X_1, \dots, X_n$ on $\mathscr{H} \otimes \Gamma$: let $G_j(X_1, \dots, X_n, t)$ $(j = 1, 2, \dots, n)$ be odd, self-adjoint, operator-valued functions. If F and G are adapted and obey Lipschitz conditions, then the system

$$\mathrm{d}X_j(t) = \sum_k F_{jk}(X_1(t), \dots, X_n(t), t)\,\mathrm{d}\Psi_k(t) + G_j(X_1(t), \dots, X_n(t), t)\,\mathrm{d}t \tag{20}$$

has a unique (odd) adapted solution $(X_1(t), \dots, X_n(t))$ for each (odd) initial value $(X_1(0), \dots, X_n(0))$ acting on $\mathscr{H}$. The proof of this is a slight modification of Barnett *et al* (1983) and we omit it. We choose $X_j(0) = \psi_j$, the Clifford elements on $\mathscr{H}$, which obey

$$\psi_j\psi_k + \psi_k\psi_j = 2\delta_{jk} \qquad (\mathrm{CAR}). \tag{21}$$

It is possible to choose F, G so that the solution $X_j(t)$ obeys the CAR for all time. Suppose this to be the case. Then for each t there is a unique C^*-algebra isomorphism τ_t from the Clifford algebra $\mathscr{C}$ generated by $\psi_1, \dots, \psi_n$ onto the Clifford algebra generated by $X_1(t), \dots, X_n(t)$, such that $\tau_t\psi_j = X_j(t)$ for each t. A function of $\psi_1, \dots, \psi_n$, that is, an operator $f \in \mathscr{C}$, can then be made to evolve in time by $f_t = \tau_t f$. Such a function has a unique expansion

$$f = a_0 1 + \sum_j a_j\psi_j + \sum_{j<k} a_{jk}\psi_j\psi_k + \dots + a_{1\dots n}\psi_1 \dots \psi_n.$$

This leads to

$$f_t = a_0 1 + \sum_j a_j X_j(t) + \sum_{j<k} a_{jk} X_j(t) X_k(t) + \cdots + a_{1\dots n} X_1(t) \dots X_n(t).$$

We can then unambiguously calculate

$$\begin{aligned} \mathrm{d}f_t(X_1(t),\ldots,X_n(t)) &= f_t(X_1+\mathrm{d}X_1,\ldots,X_n+\mathrm{d}X_n)-f_t(X_1,\ldots,X_n) \\ &= \sum_j (\partial f/\partial X_j)\,\mathrm{d}X_j + \sum_{j<k} (\partial^2 f/\partial X_j\,\partial X_k)\,\mathrm{d}X_j\,\mathrm{d}X_k \end{aligned}$$

up to second order, where the formal symbols $\partial/\partial X_j$, $\partial^2/\partial X_j\,\partial X_k$ are introduced to stand for the coefficients of $\mathrm{d}X_j$ and $\mathrm{d}X_j\,\mathrm{d}X_k$. In popular parlance, f is a 'superfield'. We now substitute for $\mathrm{d}X_j$ from (20), and integrate it from s to t. This yields

$$f(X(t))-f(X(s)) = \int_s^t \mathscr{L}f(X(\tau))\,\mathrm{d}\tau + [\text{stochastic integral}]$$

where $\mathscr{L}$ is the 'second-order differential operator'

$$f \mapsto \sum_j (\partial f/\partial X_j) G_j + \sum_{j<k} (\partial^2 f/\partial X_j\,\partial X_k) \sum_l F_{jl}F_{lk}$$

and the [stochastic integral] is given by $\int_s^t \Sigma_{j,k}(\partial f/\partial X_j)F_{jk}\,\mathrm{d}\Psi_k$. Since the stochastic integral is a martingale, it drops out when we condition onto time s: putting $s=0$, we get

$$\mathbb{E}\{f(X(t))|\text{time}=0\} = f(\psi) + \int_0^t \mathbb{E}\mathscr{L}\{f(X(\tau))|0\}\,\mathrm{d}\tau, \tag{22}$$

i.e. the left-hand side solves the 'diffusion' equation $\partial f/\partial t = \mathscr{L}f$ in superspace. A satisfactory answer to the corresponding martingale problem, converse to the above, requires a more detailed analysis which must be postponed.

Nevertheless, the above result generalises the work of Applebaum and Hudson (1983), who treat the linear case with one degree of freedom, in which case the second derivative cannot occur. Our solution exhibits some of the structure desired by Frigerio and Gorini (1983), while avoiding the stationarity and the limit needed there.

While this manuscript was being prepared, we learnt of the more general set of examples given by the method of R L Hudson and K Parthasarathy: we would like to thank them for some very useful discussions. We are indebted to J R Klauder and T Yulick at the Bell Laboratories for a quick preparation of the manuscript.

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